QUIZ 1 - CALCULUS 3 (2021/3/11)

1. Find partial derivatives of the following functions.

   (a) (6 pts) Find f_x, f_y and f_z, where $f(x, y, z) = (x^2 + y^2)^{yz}$.

   (b) (4 pts) Find f_x and f_{xy}, where $f(x, y) = \arctan(\frac{2y}{x})$.

   Solution:

   (a) $f_x = yz(x^2 + y^2)^{yz-1} \cdot 2x = 2xyz(x^2 + y^2)^{yz-1}$. (2 pts)

   $\ln(f) = yz \ln(x^2 + y^2)$. Hence $\frac{f_y}{f} = z \ln(x^2 + y^2) + yz\frac{2y}{x^2 + y^2}$ which implies that

   $f_y = f \cdot (z \ln(x^2 + y^2) + \frac{2y^2 z}{x^2 + y^2}) = (x^2 + y^2)^{yz}(z \ln(x^2 + y^2) + \frac{2y^2 z}{x^2 + y^2})$. (2 pts)

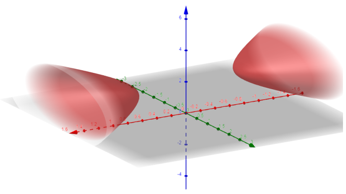
   $f_z = \ln(x^2 + y^2)(x^2 + y^2)^{yz} \cdot y = y \ln(x^2 + y^2)(x^2 + y^2)^{yz}$. (2 pts)

   (b) $f_x = \frac{1}{1 + (\frac{2y}{x})^2} \cdot \frac{-2y}{x^2} = \frac{-2y}{x^2 + 4y^2}$. (2 pts)

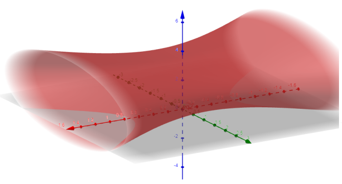
   $f_{xy} = (f_x)_y = \frac{-2}{x^2 + 4y^2} - \frac{-2y \cdot 8y}{(x^2 + 4y^2)^2} = \frac{-2x^2 + 8y^2}{(x^2 + 4y^2)^2}$. (2 pts)

2. Consider the surface $S : -2x^2 + y^2 + z^2 - 4z = -1$. Near the point $(1, 1, 4)$, S is the graph of an implicit function $z = f(x, y)$.

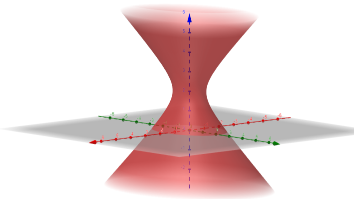
   (a) (2 pts) Choose the graph of S.



A



B



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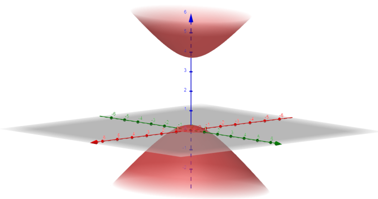

D

   (b) (4 pts) Use the implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for all $x, y, z \neq 2$.

   (c) (2 pts) Find an equation of the tangent plane to S at $(1, 1, 4)$.

   (d) (2 pts) Use the linear approximation of $f(x, y)$ at $(1, 1)$ to estimate $f(1.01, 0.98)$.

   Solution:

   (a) The graph of S is B.

   (b) Method 1:

   Differentiate the equation $-2x^2 + y^2 + z(x, y)^2 - 4z(x, y) = -1$ with respect to x. We obtain $-4x + 2z \cdot \frac{\partial z}{\partial x} - 4\frac{\partial z}{\partial x} = 0$. Hence $\frac{\partial z}{\partial x} = \frac{2x}{z - 2}$. (2 pts)

Differentiate the equation $-2x^2+y^2+z(x,y)^2-4z(x,y)=-1$ with respect to y. We obtain $2y+2z\cdot\frac{\partial z}{\partial y}-4\frac{\partial z}{\partial y}=0$. Hence $\frac{\partial z}{\partial x}=\frac{-y}{z-2}$. (2 pts)

Method 2: Near the point $(1,1,4)$, we can solve for z. $z=2+\sqrt{3+2x^2-y^2}$. Hence $\frac{\partial z}{\partial x}=\frac{2x}{\sqrt{3+2x^2-y^2}}$, and $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{3+2x^2-y^2}}$.

(c) From part (b) we know that $\frac{\partial f}{\partial x}(1,1)=\frac{2\cdot 1}{4-2}=1$ and $\frac{\partial f}{\partial y}(1,1)=\frac{-1}{4-2}=-\frac{1}{2}$. The tangent plane of S at $(1,1,4)$ is

$$z=f(1,1)+\frac{\partial f}{\partial x}(1,1)\cdot(x-1)+\frac{\partial f}{\partial y}(1,1)\cdot(y-1)=4+(x-1)-\frac{1}{2}(y-1)=x-\frac{1}{2}y+\frac{7}{2}.$$

(d) By the definition of linear approximation,

$$f(1.01,0.98)\approx 4+(1.01-1)-\frac{1}{2}(0.98-1)=4.02.$$